Time Series lecture 11

Linear time-invariant filters

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Lecture outline

1. Linear filters
2. Impulse response and transfer function
   - Impulse response
   - Transfer function
3. The spectral density function and linear filters

Linear filters

Digital filters

A digital filter maps some input sequence to another output sequence. In this lecture, these sequences will have the index set $\mathbb{Z}$. Other filters exist, but in this course, filter will mean digital filter.

- If a filter is given by L, then the filter of a sequence $\{X_t\}$ will be another sequence, say $\{Y_t\}$, and we write

$$\{Y_t\} = L\left[\{X_t\}\right]. \tag{11.1}$$

- When we refer to an element of the output of a filter, we will write a subscript after the bracket, e.g.

$$Y_u = L\left[\{X_t\}\right]_u. \tag{11.2}$$

To be clear, the t in $\{X_t\}$ is a dummy variable, whilst $u \in \mathbb{Z}$ has meaning.

We have dropped the index set from the sequences for notational convenience, i.e. in this lecture $\{X_t\} = \{X_t\}_{t\in\mathbb{Z}}$.

Linear time-invariant filters

Recall the backshift operator, denoted B. It has the property that

$$\mathrm{B}\left[\{X_t\}\right]_u = X_{u-1} \tag{11.3}$$

for any $u \in \mathbb{Z}$.

Definition 11.1

A digital filter L is called a linear time-invariant (LTI) filter if for any sequences $\{X_t\}$, $\{Y_t\}$ and any $\alpha \in \mathbb{C}$ we have

1. Linearity:

$$L\left[\{\alpha X_t + Y_t\}\right] = \alpha L\left[\{X_t\}\right] + L\left[\{Y_t\}\right]. \tag{11.4}$$

2. Time invariance:

$$L\left[\mathrm{B}\left[\{X_t\}\right]\right] = \mathrm{B}\left[L\left[\{X_t\}\right]\right]. \tag{11.5}$$

Some examples

Example 11.2

The backshift operator is clearly a linear time-invariant filter.

Example 11.3

Consider a filter L such that for any $u \in \mathbb{Z}$

$$L\left[\{X_t\}\right]_u = X_u + \phi X_{u-1},$$

where $\phi \in \mathbb{R}$. This is linear because for any $u \in \mathbb{Z}$

$$\begin{aligned} L\left[\{\alpha X_t + Y_t\}\right]_u &= \alpha(X_u + \phi X_{u-1}) + Y_u + \phi Y_{u-1} \\ &= \alpha L\left[\{X_t\}\right]_u + L\left[\{Y_t\}\right]_u . \end{aligned}$$

Time invariance is left to the interested reader, but is easily verified.

Arbitrary time shifts

Say we want to shift by some arbitrary $u \in \mathbb{Z}$, then we need only apply B u times, i.e. for any $s \in \mathbb{Z}$

$$\mathrm{B}^u \left[\{X_t\}\right]_s = X_{s-u}. \tag{11.6}$$

(If $u < 0$, this means applying the inverse operator, B^{-1}, $-u$ times).
If L is an LTI filter, then for any $u \in \mathbb{Z}$,

$$L\left[\mathrm{B}^u \{X_t\}\right] = \mathrm{B}^u L\left[\{X_t\}\right]. \tag{11.7}$$

Often, (11.7) is stated as the condition for time-invariance (as this parallels the continuous-time version), however, one only needs to show the weaker condition given in (11.5).

Linear combinations of linear filters

Proposition 11.4

Consider two LTI filters L_1 and L_2, and let $\alpha \in \mathbb{C}$ then the filter

$$L = \alpha L_1 + L_2 \tag{11.8}$$

is also an LTI filter.

Proposition 11.5

Consider two LTI filters L_1 and L_2. The filter $L = L_1 L_2$, i.e. so that

$$L[\{X_t\}] = L_1[L_2[\{X_t\}]] \tag{11.9}$$

is also an LTI filter. (This is called a cascaded filter.)

- See the exercises for proofs of these two results.

Impulse response and transfer function

Properties of a filter

In order to explore the properties of a given LTI filter, we will consider the effect of applying the filter to some test sequences. The sequences in question will be:

1. the impulse sequence,
2. a complex wave.

Formal definitions will be given on the following slides, however, informally they tell us about:

1. the effect of the filter if we input a single shock to the system at a given time,
2. the effect the filter has on a signal made up of one specific frequency.

The impulse response sequence

For a given $m \in \mathbb{Z}$, define the impulse sequence $\{\delta_{t,m}\}$ so that for $t \in \mathbb{Z}$

$$\delta_{t,m} = \begin{cases} 1 & \text{if } m = t, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 11.6

For some LTI filter L, let the impulse response sequence be $\{h_m\}$ such that for any $m \in \mathbb{Z}$

$$h_m = L\left[\{\delta_{t,-m}\}\right]_0 . \tag{11.10}$$

- Note: the zero here is the time index.

Linear filters as a convolution

Theorem 11.7

A digital filter L is an LTI filter if and only if we can write the filter output as a convolution:

$$L\left[\{X_t\}\right]_u = \sum_{m\in\mathbb{Z}} h_{u-m} X_m \tag{11.11}$$

for any $u \in \mathbb{Z}$.

- The $\{h_m\}$ here is the same impulse response we defined on the previous slide.
- The proof of this result will be in the exercises.

The transfer function

Recall, the second kind of test sequence of interest is a complex wave. For $f \in \mathbb{R}$, let $\left\{\xi_{t,f}\right\}$ be such that for all $t \in \mathbb{Z}$,

$$\xi_{t,f} = e^{2\pi i f t}.$$

Definition 11.8

The transfer function of an LTI filter L is

$$H(f) = L\left[\left\{\xi_{t,f}\right\}\right]_0, \qquad f \in \mathbb{R}. \tag{11.12}$$

- The transfer function is useful for understanding the frequency domain properties of a linear filter.

Theorem 11.9

Consider an LTI filter L. The $\{\xi_{t,f}\}$ *sequences are the eigensequences and* $H(f)$ *the eigenvalues of L. In other words,*

$$L\left[\{\xi_{t,f}\}\right] = H(f)\{\xi_{t,f}\} \tag{11.13}$$

for any $f \in \mathbb{R}$.

- So we see that applying a linear filter to a complex wave just modifies its phase and amplitude, whilst retaining the same frequency.
- The shape of $H(f)$ determines how this weighting occurs.

Proof of Theorem 11.9.

For any $u \in \mathbb{Z}$ and $f \in \mathbb{R}$, begin by noting that for any $\tau \in \mathbb{Z}$

$$\mathrm{B}^{-u}\left[\left\{\xi_{t,f}\right\}\right]_{\tau} = \xi_{\tau+u,f} = e^{2\pi i(\tau+u)f} = e^{2\pi i u f}\xi_{\tau,f},$$

and so

$$\mathrm{B}^{-u}\left[\left\{\xi_{t,f}\right\}\right] = e^{2\pi i u f}\left\{\xi_{t,f}\right\}. \tag{11.14}$$

Therefore

$$\begin{aligned}
L\left[\left\{\xi_{t,f}\right\}\right]_u &= \mathrm{B}^u\left[L\left[\mathrm{B}^{-u}\left[\left\{\xi_{t,f}\right\}\right]\right]\right]_u && \text{(time invariance)} \\
&= L\left[e^{2\pi i f u}\left\{\xi_{t,f}\right\}\right]_0 && \text{(by (11.14))} \\
&= e^{2\pi i f u} L\left[\left\{\xi_{t,f}\right\}\right]_0 && \text{(linearity)} \\
&= \xi_{u,f} H(f). && \text{(by definition)}
\end{aligned}$$

Therefore since this holds for any $u \in \mathbb{Z}$, the result holds. □

Why did we do this?

If $\{X_t\}$ is second order stationary, then we have two representations

$$X_t = \sum_{m\in\mathbb{Z}} X_m \delta_{m,t}, \qquad \text{(sifting property)}$$

$$X_t \overset{ms}{=} \mu + \int_{-1/2}^{1/2} e^{2\pi i f t} \mathrm{d}Z(f). \qquad \text{(spectral representation)}$$

Thus $L[\{X_t\}]$ is

$$L[\{X_t\}]_u = \sum_{m\in\mathbb{Z}} X_m L[\{\delta_{m,t}\}]_u,$$

$$L[\{X_t\}]_u \overset{ms}{=} \mu + \int_{-1/2}^{1/2} L\left[\left\{e^{2\pi i f t}\right\}\right]_u \mathrm{d}Z(f)$$

$$= \mu + \int_{-1/2}^{1/2} H(f) e^{2\pi i f u} \mathrm{d}Z(f).$$

Relating the impulse response and transfer function

Theorem 11.10

The transfer function is the Fourier transform of the impulse response, in other words, assuming $\{h_m\} \in \ell^1$, *for* $f \in \mathbb{R}$

$$H(f) = \sum_{m \in \mathbb{Z}} h_m e^{-2\pi i f m}. \tag{11.15}$$

- This provides a convenient way to compute the transfer function.
- Notice that since the impulse response was a convolution, Theorem 11.9 is a consequence of the convolution theorem! (See exercises for details.)

Proof.

We have for any $f \in \mathbb{R}$

$$\begin{aligned}\sum_{m\in\mathbb{Z}} h_m e^{-2\pi i f m} &= \sum_{m\in\mathbb{Z}} L\left[\{\delta_{t,-m}\}\right]_0 e^{-2\pi i f m} \\ &= L\left[\left\{\sum_{m\in\mathbb{Z}} \delta_{t,-m} e^{-2\pi i f m}\right\}\right]_0 \qquad \text{(linearity)} \\ &= L\left[\left\{e^{2\pi i f t}\right\}\right]_0 \\ &= H(f).\end{aligned}$$

where the third line follows because $\delta_{t,-m}$ is one if $m = -t$ and zero otherwise. □

The spectral density function and linear filters

Spectra of the output process

In what follows, it will be convenient to augment our usual notation. For the stationary discrete-time time series $\{X_t\}_{t\in\mathbb{Z}}$, we will write

- μ_X for the mean,
- $\gamma_\tau^{(X)}$ for the autocovariance,
- $S_X(f)$ for the spectral density function.

Theorem 11.11

Consider a stationary time series $\{X_t\}_{t\in\mathbb{Z}}$. If L is an LTI filter with impulse response $h \in \ell^1$, and

$$\{Y_t\} = L[\{X_t\}]$$

then $\{Y_t\}$ is a stationary process.

Proof.

From Theorem 11.7, $Y = h * X$. Thus for any $t \in \mathbb{Z}$, by Fubini's theorem

$$\mathbb{E}[Y_t] = \sum_{m \in \mathbb{Z}} h_m \mathbb{E}[X_{t-m}] = \mu_X \sum_{m \in \mathbb{Z}} h_m$$

which does not depend on t. Now for $t, \tau \in \mathbb{Z}$ we have

$$\begin{aligned} \operatorname{Cov}(Y_t, Y_{t+\tau}) &= \sum_{m \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} h_m h_s \operatorname{Cov}(X_{t-m}, X_{t+\tau-s}) && \text{(Fubini)} \\ &= \sum_{m \in \mathbb{Z}} \sum_{s \in \mathbb{Z}} h_m h_s \gamma^{(X)}_{\tau+m-s} && \text{(stationarity)} \end{aligned}$$

which does not depend on t. Finally by stationarity and $h \in \ell^1$

$$\gamma^{(Y)}_0 \leq \gamma^{(X)}_0 \|h\|_1^2 < \infty.$$

□

Theorem 11.12

Consider a stationary time series $\{X_t\}_{t\in\mathbb{Z}}$, with $\gamma^{(X)} \in \ell^1$. If L is an LTI filter with impulse response $h \in \ell^1$, and $\{Y_t\} = L[\{X_t\}]$ then

$$S_Y(f) = |H(f)|^2 S_X(f) \tag{11.16}$$

where H is the transfer function of L.

- From this result, we can conveniently compute the spectral density function of different processes.

Proof.

Firstly from the proof of Theorem 11.11

$$\begin{aligned}
\gamma_\tau^{(Y)} &= \sum_{m\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} h_m h_s \gamma_{\tau+m-s}^{(X)} \\
&= \sum_{m\in\mathbb{Z}}\sum_{u\in\mathbb{Z}} h_m h_{u+m} \gamma_{\tau-u}^{(x)} && (u = s - m) \\
&= \sum_{u\in\mathbb{Z}}\sum_{m\in\mathbb{Z}} h_m h_{u+m} \gamma_{\tau-u}^{(x)} && \text{(Fubini)} \\
&= \sum_{u\in\mathbb{Z}} w_u \gamma_{\tau-u}^{(x)}
\end{aligned}$$

where

$$w_u = \sum_{m\in\mathbb{Z}} h_m h_{u+m}.$$

Therefore we can apply the convolution theorem, obtaining for $f \in \mathbb{R}$

$$S_Y(f) = W(f) S_X(f).$$

All that remains is to find W, the Fourier transform of w. Letting $\tilde{h}_t = h_{-t}$ for all $t \in \mathbb{Z}$, we see

$$\begin{aligned} w_u &= \sum_{m \in \mathbb{Z}} h_m h_{u+m} \\ &= \sum_{m' \in \mathbb{Z}} \tilde{h}_{m'} h_{u-m'} \qquad\qquad \text{(setting } m' = -m\text{)} \end{aligned}$$

so $w = \tilde{h} * h$ and thus applying the convolution theorem again

$$W(f) = \tilde{H}(f) H(f) = |H(f)|^2 ,$$

because $\tilde{H}(f) = H(-f) = H(f)^*$. This gives the required result. □

Lemma 11.13

The filter given by $\Phi(\mathrm{B})$ *for some* p^{th} *order polynomial* Φ *is an LTI filter with transfer function*

$$H(f) = \Phi(e^{-2\pi i f})$$

for $f \in \mathbb{R}$.

Proof.

A linear combination of LTI filters is an LTI filter. The impulse response is

$$h_m = \begin{cases} \phi_m & \text{if } 0 \leq m \leq p, \\ 0 & \text{otherwise.} \end{cases}$$

So its transfer function is

$$H(f) = \sum_{m \in \mathbb{Z}} h_m e^{-2\pi i f m} = \sum_{j=0}^{p} \phi_j e^{-2\pi i f j} = \Phi(e^{-2\pi i f}).$$

Theorem 11.14

Consider a stationary ARMA(p, q) process

$$\Phi(\mathrm{B})X_t = \theta(\mathrm{B})\epsilon_t,$$

the spectral density function of X is given by

$$\begin{aligned} S_X(f) &= \sigma^2 \frac{\left|\Theta\left(e^{-2\pi i f}\right)\right|^2}{\left|\Phi\left(e^{-2\pi i f}\right)\right|^2} \\ &= \sigma^2 \left| \frac{\sum_{j=0}^{q} \theta_j e^{-2\pi i j f}}{\sum_{j=0}^{p} \phi_j e^{-2\pi i j f}} \right|^2 \end{aligned}$$

- We do not have a nice closed form for the autocovariance function of an ARMA process.
- But, we do have nice form for the spectral density function.

Proof.

If we have a stationary ARMA process, then we can write

$$\Phi(\mathrm{B})\left[\{X_t\}\right] = \Theta(\mathrm{B})\left[\{\epsilon_t\}\right].$$

Therefore we have from Lemma 11.13

$$\left|\Phi(e^{-2\pi i f})\right|^2 S_X(f) = \left|\Theta(e^{-2\pi i f})\right|^2 S_\epsilon(f)$$

and so

$$S_X(f) = \sigma^2 \frac{\left|\Theta\left(e^{-2\pi i f}\right)\right|^2}{\left|\Phi\left(e^{-2\pi i f}\right)\right|^2}$$

because, by stationarity $\Phi\left(e^{-2\pi i f}\right) \neq 0$. □